Horn's Problem and semistability for quiver representations

based on the talk by Christof Geiß

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This is not an original work of the author, but attempt together with W. Crawley-Boevey to understand solutions of Klyachko et al. of the problem.

Denote by W_n the set of all n-tuples $\boldsymbol{\nu} := (\nu_1 \geq \cdots \geq \nu_n)$ in $\mathbb{R}^n$. Let H_n^μ be the set of all triples $(\boldsymbol{\nu}(1), \boldsymbol{\nu}(2), \boldsymbol{\nu}(3))$ from W_n^3 such that there exist Hermition matrices $H(1)$, $H(2)$, $H(3)$ with the property $\operatorname{spec}(H(s)) = \boldsymbol{\nu}(s)$ and $H(1) + H(2) + H(3) = \mu 1_n$. We want to describe the set H_n^μ.

Let k be an algebraically closed field. A quiver $Q = (Q_0, Q_1, t, h)$ can be viewed as a category. Representations of Q are functors from Q to $\operatorname{mod} k$. For a representation M of Q we may define $\mathbf{dim}\, M := (\dim_k M(x))_{x \in Q_0}$. We have Ringel form on $\mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$ such that $\langle \alpha, \beta \rangle := \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$. We also have an affine variety $\operatorname{Rep}_Q^\beta$ of representations with dimension vector β, which is by definition $\prod_{a \in Q_1} \operatorname{Hom}_k(k^{\beta(ta)}, k^{\beta(ha)})$. The group $\operatorname{Gl}_\beta := \prod_{x \in Q_0} \operatorname{Gl}_{\beta(x)}(k)$ acts on $\operatorname{Rep}_Q^\beta$ by conjugations. The orbits of this action corresponds to the isoclasses of representations of Q with dimension vector β.

Let $k[\operatorname{Rep}_Q^\beta]$ be an affine coordinate ring. We have that $\operatorname{Spec}(k[\operatorname{Rep}_Q^\beta]^{\operatorname{Gl}_\beta})$ parameterizes closed orbits in $\operatorname{Rep}_Q^\beta$. By $\operatorname{SI}(Q, \beta)$ we denote the ring of semi-invariants, that is $\operatorname{SI}(Q, \beta) = k[\operatorname{Rep}_Q^\beta]^{\operatorname{Sl}_\beta}$, where $\operatorname{Sl}_\beta := \prod \operatorname{Sl}_{\beta(x)}(k)$. We have a natural grading $\operatorname{SI}(Q, \beta) = \bigoplus_{\sigma \in \Gamma} \operatorname{SI}_\sigma(Q, \beta)$, where $\Gamma := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ and $f \in \operatorname{SI}_\sigma(Q, \beta)$ if and only if $f(gm) = \prod_{x \in Q_0} (\det g_x)^{\sigma(\varepsilon_x)} f(m)$ for $m \in \operatorname{Rep}_Q^\beta$ and $g \in \operatorname{Gl}_\beta$.

Suppose that α and β are dimension vectors such that $\langle \alpha, \beta \rangle = 0$. Let $m \in \operatorname{Rep}_Q^\alpha$. We take a projective presentation $0 \to P_1 \to P_0 \to M \to 0$ of M and the induced long exact sequence $0 \to \operatorname{Hom}_Q(M, -) \to \operatorname{Hom}_Q(P_0, -) \xrightarrow{\delta_-^M} \operatorname{Hom}_Q(P_1, -) \to \operatorname{Ext}_A^1(M, -) \to 0$. If $n \in \operatorname{Rep}_Q^\beta$ then δ_N^M is a square matrix. Thus we may define $d^m : \operatorname{Rep}_Q^\beta \to k$ by $d^m(n) := \det(\delta_N^M)$. Then $d^m \in$

$\mathrm{SI}(Q,\beta)_{\langle\alpha,-\rangle}$. Moreover $d^m(n)\neq 0$ if and only if $\mathrm{Hom}_Q(M,N)=0$. We call d^m a Schofield's semi-invariant.

Theorem (Schofield, King, 1994)**.** *Let Q be a quiver without oriented cycles and α, β two dimension vectors with $\langle\alpha,\beta\rangle=0$. The following conditions are equivalent.*

(a) *There exists a representation M with $\mathbf{dim}\, M=\beta$ and $\langle\alpha,\mathbf{dim}\, M'\rangle\leq 0$ for all submodules M' of M (we say M is $\langle\alpha,-\rangle$-semistable).*

(b) *For some $l\geq 1$ there exists $0\neq f\in\mathrm{SI}(Q,\beta)_{l\langle\alpha,-\rangle}$.*

(c) *For each general subrepresentation $\beta'\hookrightarrow\beta$ we have $\langle\alpha,\beta'\rangle\leq 0$.*

(d) $\mathrm{ext}(\alpha,\beta)=0$, *where* $\mathrm{ext}(\alpha,\beta)=0$ *is the minimum of* $\dim_k\mathrm{Ext}_Q(N,M)$ *for N and M with $\mathbf{dim}\, N=\alpha$ and $\mathbf{dim}\, M=\beta$ respectively.*

(e) *There exists $v\in\mathrm{Rep}_Q^{\alpha}$ such that $d^v:\mathrm{Rep}_Q^{\beta}\to k$ is nonzero.*

(f) *If $k=\mathbb{C}$ there exists a representation $w\in\mathrm{Rep}_Q^{\beta}$ such that for each $x\in Q_0$ we have $\sum_{\substack{a\in Q_1\\ h(a)=x}}W(a)W(a)^+-\sum_{\substack{a\in Q_1\\ t(a)=x}}W(a)^+W(a)=\langle\alpha,\varepsilon_x\rangle 1_{\mathbb{C}^{\beta(x)}}$, where A^+ denotes the conjugate transpose of A.*

Let Q be the following quiver:

$$\begin{array}{ccccccc}
x_1(1) & \leftarrow & \cdots & \leftarrow & x_{n-1}(1) & & \\
 & & & & & \nwarrow & \\
x_1(2) & \leftarrow & \cdots & \leftarrow & x_{n-1}(2) & \leftarrow & x_n \\
 & & & & & \swarrow & \\
x_1(3) & \leftarrow & \cdots & \leftarrow & x_{n-1}(3) & &
\end{array}$$

and $\beta=\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & \end{array}$. Assume that we have $\boldsymbol{\nu}(1)$, $\boldsymbol{\nu}(2)$, $\boldsymbol{\nu}(3)$ inW_n with integral coefficients and $\boldsymbol{\nu}_n(s)=0$. Assume also that $\mu=\frac{1}{n}\sum_{i,s}\nu_i(s)$ is an integer. Let $\alpha_{\boldsymbol{\nu}}$ be a dimension vector such that $\langle\alpha_{\boldsymbol{\nu}},\varepsilon_{x_i(s)}\rangle=\nu_i(s)-\nu_{i+1}(s)$ and $\langle\alpha_{\boldsymbol{\nu}},\varepsilon_{x_n}\rangle=\mu$.

Proposition (Derksen, Weyman)**.** *Number of summands isomorphic to $S_{\mu^n}(\mathbb{C}^n)$ in $\bigotimes_{s=1}^3 S_{\boldsymbol{\nu}(s)}(\mathbb{C}^n)$ equals* $\dim\mathrm{SI}(Q,\beta)_{\langle\boldsymbol{\nu},-\rangle}=\dim(\bigotimes_s S_{\boldsymbol{\nu}(s)})^{\mathrm{Sl}_n(\mathbb{C})}$.

Let $\mathscr{P}_r^n$ be a set of all r-tuples $1\leq i_1<\cdots<i_r\leq n$. For $\mathbf{I}=(\mathbf{i}(1),\mathbf{i}(2),\mathbf{i}(3))$, where $\mathbf{i}(s)\in\mathscr{P}_r^n$ we define a dimension vector $\beta_{\mathbf{I}}$ such that $\langle\alpha_{\boldsymbol{\nu}},\beta_{\mathbf{I}}\rangle=\sum_s\sum_j\nu_{i_j(s)}(s)$.

Proposition. *We have* $\beta_{\mathbf{I}} \hookrightarrow \beta$ *if and only if* $\prod_s \sigma_{\boldsymbol{\lambda}(\mathbf{i}(s))} \neq 0 \in H^*(\mathrm{Gr}_r^n(\mathbb{C}))$.

Theorem. *The following are equivalent:*

(c) *For* $1 \leq r \leq n$ *and each* $\mathbf{I} \in \mathscr{P}_r^n$ *with* $\sum_s \boldsymbol{\lambda}(\mathbf{i}(s)) = r(n-r)$ *and* $\prod \sigma_{\boldsymbol{\lambda}(\mathbf{i}(s))} \neq 0$ *we have* $\frac{1}{r} \sum_s \sum_{j=1}^r \nu_{i_j(s)}(s)) \leq \mu$.

(e) $\bigotimes_{s=1}^3 S_{\boldsymbol{\nu}(s)}(\mathbb{C}^n)$ *has a summand isomorphic to* $S_{(\mu^n)}(\mathbb{C}^n)$.

(f) *There exists Hermitian matrices* $H(1)$, $H(2)$, $H(3)$ *with* $\sum H(s) = \mu 1_n$ *and* $\mathrm{spec}(H(s)) = \boldsymbol{\nu}(s)$.